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Functional delta-functions and Fourier transforms

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Abstract. A degenerate functional integral is defined. This integral yields some asymptotic relations for functional Fourier transforms, and might furnish a mathematical basis for introducing delta functions and their derivatives. Feynman-type integrals as well as measure-theoretic integrals are included in the discussion.

1. Introduction

The Fourier transform is a basic tool in the study of finite-dimensional distributions. This transform has also been used heuristically in various analogous infinite-dimensional situations. E.g., in the path-integral treatment of the electromagnetic field, one encounters formulae like

$$\int \mathscr{D}(\lambda)\, \exp(\mathrm{i}\langle\lambda, \partial^k A_k\rangle) = \delta(\partial^k A_k). \tag{1.1}$$

This equation serves to specify the gauge (cf e.g. Faddeev and Slavnov 1980).

It would clearly be desirable to include such relations within the mathematical theory of Feynman path integrals (and of other functional integrals). The present note outlines a possible approach to this problem. In particular, we justify the formula

$$(-\mathrm{i})^j \int \mathscr{D}_1(\xi)\mathscr{D}_2(\zeta)\, \mathrm{e}^{\mathrm{i}\langle\xi,\zeta\rangle}\langle\beta_1,\zeta\rangle\ldots\langle\beta_j,\zeta\rangle f(\xi) = D_{\beta_1}\ldots D_{\beta_j} f(0). \tag{1.2}$$

The functional integral will be defined in two ways, and will be related to asymptotic limits of more familiar integrals.

A given real Hilbert space $\mathscr{H}$ is presupposed, in (1.2) and elsewhere. More precisely, we should take two such spaces $\mathscr{H}_\xi$ and $\mathscr{H}_\zeta$ with a specified isomorphism, but usually we will be ignoring this subtlety. The $\mathscr{D}$ symbols will have the usual translational invariance, $\mathscr{D}_j(\chi+\alpha) = \mathscr{D}_j(\chi)$ (for $\alpha \in \mathscr{H}$). The D_β's are differential operators,

$$D_\beta f(\xi) := \langle\beta, \delta/\delta\xi\rangle f(\xi) := (\mathrm{d}/\mathrm{d}\varepsilon)_{\varepsilon=0} f(\xi+\varepsilon\beta). \tag{1.3}$$

(We will be concerned primarily with Gâteaux derivatives like $D_\beta f$.) An empty product, e.g. for $j=0$ in (1.2), is unity.

The content of (1.2) for the case $j=0$ can be expressed as follows:

$$\int \mathscr{D}_2(\zeta)\, \exp(\mathrm{i}\langle\xi,\zeta\rangle) = \delta(\xi), \qquad \int \mathscr{D}_1(\xi)\delta(\xi) f(\xi) = f(0). \tag{1.4a, b}$$

(If $j \geq 1$, we would have derivatives of δ.) The δ function occurs here naturally, but we do not consider here the question whether (1.4*a*) or (1.4*b*) could be exploited for a general and effective definition of $\delta(\xi)$. We note, however, that the symbols $\mathscr{D}_j$ and δ can be determined by the above only up to a scale transformation, $\mathscr{D}_1 \to c\mathscr{D}_1$, $\mathscr{D}_2 \to c^{-1}\mathscr{D}_2$, $\delta \to c^{-1}\delta$, where $c > 0$.

We mention also other studies of distributions in infinitely many variables, by Kristensen *et al* (1965, 1967) and Krée (1976). In these studies no attempt was made to clarify formulae such as (1.2) or (1.4).

2. Feynman-type integrals and their asymptotic limits

Let f_μ be the Fourier transform of a bounded Borel measure μ on $\mathscr{H}$,

$$f_\mu(\xi) = \int d\mu(\chi)\, e^{i\langle\chi,\xi\rangle}. \tag{2.1}$$

Definitions of Feynman-type integrals over $\mathscr{H}$ lead (ordinarily) to the following evaluation (e.g. Itô 1966, Tarski 1979):

$$\int \mathscr{D}(\xi) \exp(\tfrac{1}{2}i\kappa\langle\xi,\xi\rangle) f_\mu(\xi) = \int d\mu(\chi) \exp[\tfrac{1}{2}(i\kappa)^{-1}\langle\chi,\chi\rangle]. \tag{2.2}$$

The restrictions $\operatorname{Im}\kappa \geq 0$, $\kappa \neq 0$ are assumed here and below. If $|\kappa| \to \infty$ then the bounded convergence theorem yields $f_\mu(0)$ for the RHS. (Indeed, taking this limit is equivalent to replacing exp[...] by $\exp[-\frac{1}{2}i\lambda\ldots]$ and letting $\lambda \to 0$.) The analysis of this section is, in essence, an elaboration of this fact.

Equation (1.2) suggests that phase-space integrals should be relevant here, and for their definition we take that of I_{ps}, as given in Tarski (1982). This article, moreover, discusses linear factors, such as in (1.2), and in this connection cf also Berg and Tarski (1981). Following the cited works, we make the hypothesis

$$\int d|\mu|(\chi)(1 + |\langle\beta_1,\chi\rangle|)\ldots(1 + |\langle\beta_j,\chi\rangle|) < \infty, \tag{2.3}$$

and then (cf proposition 4 of Tarski (1982)),

$$\begin{aligned}(-i)^j \int \mathscr{D}_1(\xi)\mathscr{D}_2(\zeta)\, e^{i\langle\xi,\zeta\rangle} \exp[\tfrac{1}{2}(i\kappa)^{-1}\langle\zeta,\zeta\rangle]\langle\beta_1,\zeta\rangle\ldots\langle\beta_j,\zeta\rangle f_\mu(\xi) \\ = \int d\mu(\chi)\{\langle\beta_1,\delta/\delta\zeta'\rangle\ldots\langle\beta_j,\delta/\delta\zeta'\rangle(-1)^j e^{-i\langle\chi,\zeta'\rangle}\}_{\zeta'=0} \exp[\tfrac{1}{2}(i\kappa)^{-1}\langle\chi,\chi\rangle].\end{aligned} \tag{2.4}$$

(We used here an alternative notation for $I_{ps}(f_\mu\ldots)$.) A heuristic passage to the limit $|\kappa| \to \infty$ now yields (1.2), and we should like to give a precise description of the situation.

We start by adapting the definition of the integral I_{ps} to this limiting case. In particular, we use finite-dimensional projections P, with $P\xi = u$, $P\zeta = v$, $F_P = F(P\cdot, P\cdot)$, and convergence factors (let $\dim P = k$):

$$\begin{aligned}I_P^{b,\alpha,b_0,\alpha_0}(F) = [(bb_0+1)/2\pi]^k \int d^k u\, d^k v \exp(-\tfrac{1}{2}b\langle u - P\alpha, u - P\alpha\rangle) \\ \times \exp(-\tfrac{1}{2}b_0\langle v - P\alpha_0, v - P\alpha_0\rangle)\, e^{i\langle u,v\rangle} F_P(u,v).\end{aligned} \tag{2.5}$$

Then we take limits, first as $P \to 1$ by following a sequence belonging to a determining family, and then as b, $b_0 \to 0$ non-tangentially, requiring independence of α, α_0 in this latter limit, as in Tarski (1982). We arrive in this way at an integral which we denote by I_{ps}^{∞}. We will also write

$$I_{ps}^{\infty}(F) = \int \mathscr{D}_1(\xi)\mathscr{D}_2(\zeta)\, e^{i\langle\xi,\zeta\rangle} F(\xi, \zeta). \tag{2.6}$$

The invariance properties of this integral follow immediately from the construction. We identify $\mathscr{H}$ with $\mathscr{H}_\xi$ and with $\mathscr{H}_\zeta$, as before.

Proposition 1. Let α_1, $\alpha_2 \in \mathscr{H}$, and let R be an orthogonal transformation on $\mathscr{H}$. Let F be integrable for I_{ps}. Then

$$I_{ps}^{\infty}(F) = \int \mathscr{D}_1(\xi)\mathscr{D}_2(\zeta) \exp(i\langle\xi+\alpha_1, \zeta+\alpha_2\rangle) F(\xi+\alpha_1, \zeta+\alpha_2) \tag{2.7a}$$

$$= \int \mathscr{D}_1(\xi)\mathscr{D}_2(\zeta)\, e^{i\langle\xi,\zeta\rangle} F(R\xi, R\zeta). \tag{2.7b}$$

(The families of sequences of projections which yield convergence are as in proposition 3 of Tarski (1982).)

We restate the last sentence more fully. Both members of (2.7*a*) converge with reference to the same determining family (of sequences of projections), while for (2.7*b*) we have to transform this family by a rotation. Now, the following property of this integral is central for us.

Proposition 2. Let μ satisfy (2.3) and let f_μ be its Fourier transform. Then $\langle\beta_1, \zeta\rangle \ldots \langle\beta_j, \zeta\rangle f_\mu(\xi)$ is integrable for I_{ps}^{∞}, and (1.2) is fulfilled by this integral. Furthermore, the two members of (2.4) approach those of (1.2) as $|\kappa| \to \infty$. (The reference family of sequences of projections is the maximal one.)

The following proof follows closely that of proposition 4 of Tarski (1982), as given in the appendix.

Proof. We first note that the factors $\langle\beta_k, \zeta\rangle$ can be replaced by having $i^{-1}\langle\beta_k, \delta/\delta\varphi\rangle$ act on $e^{i\langle\zeta,\varphi\rangle}$, with $\varphi \to 0$ at the end. (The assumed bound on μ justifies the relevant interchanges of operations.) We next restrict ourselves to the exponential in (2.1), with the integration to be considered later. Thus, let $F_0 = \exp(i\langle\chi, \xi\rangle + i\langle\varphi, \zeta\rangle)$. We can compute $I_P^{b,\ldots}(F_0)$ explicitly, and after letting $P \to 1$ we obtain

$$I^{b,\alpha,b_0,\alpha_0}(F_0) = \exp[-\tfrac{1}{2}b\langle\alpha, \alpha\rangle + i\langle\alpha_0, \varphi\rangle - \tfrac{1}{2}b_0^{-1}\langle\varphi, \varphi\rangle]$$

$$\times \exp[-\tfrac{1}{2}B(b, b_0)\langle\chi+\psi, \chi+\psi\rangle], \tag{2.8a}$$

$$B(b, b_0) = b_0(bb_0+1)^{-1}, \qquad \psi = ib_0^{-1}\varphi - ib\alpha + \alpha_0. \tag{2.8b, c}$$

(Here the scalar product is bilinear and symmetric, but not Hermitian.)

Let us determine a bound for the last exponential. Note that B is Fréchet differentiable at $(0, 0)$, and that $B(0, 0) = \partial_b B(0, 0) = 0$, $\partial_{b_0} B(0, 0) = 1$. Therefore we have for small b, b_0 (this restriction is to be assumed below as well)

$$-\varepsilon(b, b_0)(|b| + |b_0|) + \mathrm{Re}\, b_0 \leq \mathrm{Re}\, B(b, b_0), \tag{2.9}$$

where ε is some non-negative function which $\to 0$ as b, $b_0 \to 0$. Moreover, in view of the non-tangentiality we may suppose $c|b_0| \leq \text{Re } b_0$ for some $c > 0$. From these considerations we obtain a bound of the form

$$|\exp[-\tfrac{1}{2}B(b, b_0)\langle\chi, \chi\rangle]| \leq \exp(-\tfrac{1}{2}h|b_0|\|\chi\|^2) \tag{2.10}$$

for some $h > 0$. Moreover, the expression for B shows that $\exp[-B(b, b_0)\langle\chi, \psi\rangle]$ has a bound of the form $\exp(h|b_0|\|\chi\|\|\psi'\|)$ for some vector ψ', and the existence of a uniform bound of $I^{b\ldots}$ in (2.8) follows.

The bound just deduced clearly applies also to $I_P^{b,\ldots}(F_0)$. Thus one can interchange integration over χ with $P \to 1$ and with b, $b_0 \to 0$. Moreover, when these limits are applied to $I_P^{b,\ldots}(F_0)$, one obtains $e^{i\langle\varphi,\chi\rangle}$, and we recall the differential operators $\langle\beta_k, \delta/\delta\varphi\rangle$. So, after setting $\varphi = 0$, (1.2) follows.

The second part, concerning $|\kappa| \to \infty$, follows by applying the bounded convergence theorem to (2.4), as after (1.2).

Note that the second part is in effect a statement about interchangeability of limits:

$$\lim_{|\kappa|\to\infty} \lim_{b,b_0\to 0} \lim_{P\to 1} \ldots = \lim_{b,b_0\to 0} \lim_{P\to 1} \lim_{|\kappa|\to\infty} \ldots \tag{2.11}$$

We also remark that a slight generalisation of the foregoing proposition can be easily given, so as to allow linear factors $\langle\xi_1, \xi\rangle \ldots \langle\xi_l, \xi\rangle$ in the integrand. Compare with proposition 4 of Tarski (1982).

3. Function spaces and measure-theoretic integrals

In this section we give an alternative definition of the integral in (1.2). This definition will allow us to extend (1.2) to a larger class of functions f.

We first define some classes of functions on $\mathscr{H}$. Let μ be a Borel measure on $\mathscr{H}$, and let

$$\rho_n(\mu) := \int d|\mu|(\xi)(1 + \|\xi\|)^n. \tag{3.1}$$

The condition $\rho_j(\mu) < \infty$ is more restrictive than (2.3), but it is independent of the choice of vectors β_k. Let $\mathscr{F}^{(n)} = \{f_\mu : \rho_n(\mu) < \infty\}$. Thus proposition 2 applies *a fortiori* to $f_\mu \in \mathscr{F}^{(j)}$.

Let $\mathscr{C}_B^{(0)}$ be the class of bounded and continuous functions on $\mathscr{H}$, continuity referring to the strong topology, and let $\mathscr{C}_B^{(n)}$ be the class of functions having continuous and bounded (Gâteaux) derivatives of orders $l \leq n$. We also consider the $\mathscr{T}$ topology of Gross (1960, 1963), where the basic neighbourhoods of ξ are as follows:

$$\{\chi \in \mathscr{H}: \|A(\chi - \xi)\| < \varepsilon \text{ for some Hilbert–Schmidt } A\}. \tag{3.2}$$

Let $\mathscr{T}^{(0)}$ be the class of bounded and uniformly $\mathscr{T}$-continuous functions on $\mathscr{H}$. These functions are integrable for the (properly extended) isotropic Gaussian measures. Further, let $\mathscr{T}^{(n)}$ be the class of functions having bounded and uniformly $\mathscr{T}$-continuous derivatives of orders $l \leq n$. We have (cf Gross (1963), for the first inclusion)

$$\mathscr{F}^{(n)} \subset \mathscr{T}^{(n)} \subset \mathscr{C}_B^{(n)}. \tag{3.3}$$

It is also natural to set $\mathscr{F}^\infty = \bigcap_n \mathscr{F}^{(n)}$, etc, but such spaces will play no role here.

We return to the integral in (1.2). We now adapt the definition by analytic continuation in Tarski (1982). We start with a measure-theoretic integral over an appropriate extension of $\mathcal{H}_\xi + \mathcal{H}_\zeta$, in the following form:

$$J(b, \alpha; b_0, \alpha_0; F) = \int \mathcal{D}_1(\xi)\mathcal{D}_2(\zeta) \exp(-\tfrac{1}{2})\langle \xi-\alpha, \xi-\alpha\rangle \times e^{i\langle \xi,\zeta\rangle} \exp(-\tfrac{1}{2}b_0\langle \zeta-\alpha_0, \zeta-\alpha_0\rangle)F(\xi, \zeta). \tag{3.4}$$

In view of the factor $e^{i\langle \xi,\zeta\rangle}$, this integral is to be determined as an iterated integral. For the two alternative iterations, the successive variances are b_0^{-1} and $(b+b_0^{-1})^{-1}$ in one case, and b^{-1} and $(b_0+b^{-1})^{-1}$ in the other.

Such integrals can be readily handled for functions in $\mathcal{F}^{(n)}$, also with linear factors, cf Tarski (1982). For more general functions in $\mathcal{T}^{(n)}$, various unsolved problems arise. E.g. the first integration, say over ζ, defines the transform

$$\int \mathcal{D}_2(\zeta) \exp(-\tfrac{1}{2}b_0\langle \zeta, \zeta\rangle)\, e^{i\langle \zeta,\xi\rangle} f(\zeta) = \hat{f}(\xi) \exp(-\tfrac{1}{2}b_0^{-1}\langle \xi, \xi\rangle). \tag{3.5}$$

This transform resembles that of Segal (1956), but unlike the latter, it is not unitary. It appears, indeed, that at present little can be said about $\hat{f}$ for general $f \in \mathcal{T}^{(n)}$. It is also not clear if Segal's transform would be more helpful here. However, if F in (3.4) is such that $F(\xi, \zeta) = F(\xi, P\zeta)$ or $= F(P\xi, \zeta)$ for some finite-dimensional projection P, then we can first integrate over the cylindrical variable, and the difficulties associated with iteration disappear. We therefore confine our subsequent discussion to such F's. We also drop the condition made in Tarski (1982), that the two iterations yield the same result, and denote the integral so limited by J'.

Now, the absence of oscillatory Gaussian factors in (1.2) allows us to bypass analytic continuation, and to take the limit directly of measure-theoretic integrals, as b, $b_0 \searrow 0$. We therefore give $\lim(b, b_0\searrow 0)J'$ for the second definition of the integral in (1.2), make the provision that the result be independent of α, $\alpha_0 \in \mathcal{H}$, and use the following notation:

$$J'^{\infty}_{ps}(F) = \lim(b, b_0\searrow 0)J'(b, \alpha; b_0, \alpha_0; F). \tag{3.6}$$

Let us investigate the integral J'^{∞}_{ps}, with $F = f(\xi)\Pi_1^j\langle\beta_k, \zeta\rangle$ and $f \in \mathcal{T}^{(j)}$. As we specified above, the ζ integral should be done first. Our procedure now is essentially the same as it would be for the case of finite dimensionality. We replace each $\langle\beta_k, \zeta\rangle$ by $i^{-1}\langle\beta_k, \delta/\delta\xi\rangle$, we take these operators outside the ζ integral, and we evaluate the latter integral, obtaining

$$J'(b, \alpha; b_0, \alpha_0; F) = \int \mathcal{D}_1(\xi) \exp(-\tfrac{1}{2}b\langle \xi-\alpha, \xi-\alpha\rangle)f(\xi) \times i^{-j} D_{\beta_j} \ldots D_{\beta_1} \exp(i\langle \xi, \alpha_0\rangle - \tfrac{1}{2}b_0^{-1}\langle \xi, \xi\rangle). \tag{3.7}$$

Next we integrate by parts. For the integral at hand, this can be justified by factorising the Gaussian measure. E.g. for D_{β_j}, we factorise with reference to the decomposition $\mathcal{H} = [\beta_j] + [\beta_j]^\perp$. The result is

$$\begin{aligned} J &= (-i)^{-j} \int \mathcal{D}_1(\xi)[D_{\beta_1} \ldots D_{\beta_j} \exp(-\tfrac{1}{2}b\langle \xi-\alpha, \xi-\alpha\rangle)f(\xi)] \exp(\ldots) \\ &= (-i)^{-j} \exp(-\tfrac{1}{2}b\langle \alpha, \alpha\rangle) \int \mathcal{D}_1(\xi) \exp[-\tfrac{1}{2}(b+b_0^{-1})\langle \xi, \xi\rangle] \\ &\quad \times [D_{\beta_1} \ldots D_{\beta_j} f(\xi)] \exp(b\langle \xi, \alpha\rangle + i\langle \xi, \alpha_0\rangle) + O(b). \end{aligned} \tag{3.8}$$

We make the change of variable $\xi' = (b + b_0^{-1})^{1/2}\xi$. The hypothesis allows us to take $\lim(b, b_0 \searrow 0)$ inside the integral and we want to take it also inside of f and exp, while in the latter places $(b + b_0^{-1})^{-1/2}\xi' \to 0$. The latter interchanges can be justified by verifying that the convergence in theorem 1 of Gross (1960) is in this case uniform in b, b_0. However, we omit the details. There remains the normalised integral of 1, which equals 1. Therefore

$$J'^{\infty}_{ps}(F) = (-\mathrm{i})^{-j} D_{\beta_1} \ldots D_{\beta_j} f(0). \tag{3.9}$$

Moreover, translational invariance of J'^{∞}_{ps} is immediate, as before, and we summarise.

Proposition 3. (a) The integral J'^{∞}_{ps} is translationally invariant, i.e. satisfies the analogue to (2.7*a*). (b) If $f \in \mathscr{T}^{(j)}$, then the integral in (1.2) converges as J'^{∞}_{ps}, and this equation is fulfilled.

(We do not consider rotational invariance, in order to avoid discussing measure-theoretic subtleties.)

We conclude with two remarks concerning technical details. First: the constructions of I^{∞}_{ps} and of J'^{∞}_{ps} are similar, but the difference between taking b and b_0 complex for I^{∞}_{ps} and real for J'^{∞}_{ps} is basic. Indeed, given e.g. $f \in \mathscr{T}^{(0)}$, it would follow from Vitali's theorem that $I^{\infty}_{ps}(f)$ exists (and necessarily equals $J'^{\infty}_{ps}(f)$), provided the approximations with Re b, $b_0 \searrow 0$ and P finite-dimensional are uniformly bounded. But the existence of such a bound in the general case remains unknown. Second: it is an open problem, if the integral in (1.2) can be defined, and the equation established, for general $f \in \mathscr{C}^{(j)}_{\mathrm{B}}$.

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